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Linearizable planar differential systems via the inverse integrating factor

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Abstract

Our purpose in this paper is to study when a planar differential system polynomial in one variable linearizes in the sense that it has an inverse integrating factor which can be constructed by means of the solutions of linear differential equations. We give several families of differential systems which illustrate how the integrability of the system passes through the solutions of a linear differential equation. At the end of the work, we describe some families of differential systems which are Darboux integrable and whose inverse integrating factor is constructed using the solutions of a second–order linear differential equation defining a family of orthogonal polynomials.

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1. Introduction

In this work we consider planar polynomial differential systems as:

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ belong to the ring of real polynomials in one variable and are analytic in the other variable, that is, they belong to the ring $\mathbb{R}(x)[y]$ if we choose y as the variable in which they are polynomial. We will always assume that $P(x, y)$ and $Q(x, y)$ are coprime polynomials with respect to y. We denote by d the maximum of the degrees of P and Q as polynomials in y.

We define the *orbital equation* associated with system (1) as the ordinary differential equation which is satisfied by the orbits of the system, that is, the orbital equation associated

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with system (1) is either

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x, y)}{P(x, y)} \qquad \text{or} \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{P(x, y)}{Q(x, y)}.$$

The aim of this work is to study when a system (1) linearizes.

Definition 1. *We say that system (1) linearizes, or that it is linearizable, if it has an inverse integrating factor which can be constructed by means of the solutions of linear differential equations.*

We recall that the classical definition that a system (1) is linearizable is that there exists a change of variables which transforms the orbital equation associated to system (1) into a linear differential equation. The techniques used to find such a change usually come from the Lie group theory, see [3, 9] and the references therein. We do not treat this problem in this paper but the examples that we study show that there is a connection between both definitions of linearizability.

This paper is related to the integrability problem which is defined as the problem of finding a first integral for a planar differential system and determining the functional class it must belong to. We recall that a first integral $H(x, y)$ of system (1) is a function of class $\mathcal{C}^1$ in some open set $\mathcal{U}$ of $\mathbb{R}^2$, non-locally constant and which satisfies the following partial differential equation:

$$P(x, y)\frac{\partial H}{\partial x}(x, y) + Q(x, y)\frac{\partial H}{\partial y}(x, y) \equiv 0.$$

An inverse integrating factor of system (1) is a function $V(x, y)$ of class $\mathcal{C}^1$ in some open set $\mathcal{U}$ of $\mathbb{R}^2$, non-locally null and which satisfies the following partial differential equation:

$$P(x, y)\frac{\partial V}{\partial x}(x, y) + Q(x, y)\frac{\partial V}{\partial y}(x, y) = \left(\frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)\right) V(x, y).$$

The function $(\partial P/\partial x) + (\partial Q/\partial y)$ is called the *divergence* of system (1) and it is denoted by div throughout the rest of the paper. We note that the function $1/V(x, y)$ is an integrating factor for system (1) in $\mathcal{U}$, and that given an inverse integrating factor defined in $\mathcal{U}$, a first integral in $\mathcal{U} - \{V = 0\}$ can be constructed by means of the following line integral:

$$H(x, y) = \int_{(x_0, y_0)}^{(x,y)} \frac{Q(x, y)\,\mathrm{d}x - P(x, y)\,\mathrm{d}y}{V(x, y)},$$

where (x_0, y_0) is any chosen base point in $\mathcal{U}$ with $V(x_0, y_0) \neq 0$. We note that this function $H(x, y)$ is well-defined, in general, only in a simply-connected subset of $\mathcal{U} - \{V = 0\}$. In nonsimply-connected subsets of $\mathcal{U} - \{V = 0\}$, $H(x, y)$ can be a multivalued function but it continues to exhibit the dynamic behavior of the orbits in the set.

The integrability of system (1) is given, in many occasions, by the existence of invariant curves. We say that a $\mathcal{C}^1$ function $f : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ is an *invariant curve* for a system (1) if it is not locally constant and satisfies

$$P(x, y)\frac{\partial f}{\partial x}(x, y) + Q(x, y)\frac{\partial f}{\partial y}(x, y) = k_f(x, y) f(x, y),$$

with $k_f(x, y)$ a polynomial in y of degree lower or equal to $d - 1$, where d is the degree of the system in y, and it is of class $\mathcal{C}^1$ in the other variable. In the case $f(x, y)$ is a polynomial we say that $f(x, y) = 0$ is an *invariant algebraic curve* for system (1). The construction of inverse integrating factors or analytic first integrals inside certain functional classes (polynomial, rational, elementary or Liouvillian) is strongly related with the existence of invariant algebraic curves (see for instance work [8] and specially the references therein) and it belongs to the context of the Darboux theory of integrability, see [6]. When considering the integrability problem we are also addressed to study how the existence of a first integral in a certain functional class implies the existence of an inverse integrating factor inside a certain given class of functions. In the particular case that system (1) is polynomial, we have that the existence of an elementary first integral implies the existence of an inverse integrating factor which is a rational function up to a rational power. Moreover, when system (1) is polynomial, we have that the existence of a Liouvillian first integral implies the existence of an inverse integrating factor of Darboux type, see [6, 8] and the references therein for the proof of these results. These results suggest that the functional class of an inverse integrating factor is usually easier than the functional class of a first integral. This is the reason why we look for an inverse integrating factor to study the integrability of system (1). Moreover, the inverse integrating factor is shown to be defined in phase portraits in which the dynamics avoids the existence of a first integral. However, there are also systems whose dynamics avoid the existence of an inverse integrating factor, see [5].

In the work [8], systems of the form (1) whose integrability is given by the solutions of linear differential equations are described. We obtained a result which allows to find an explicit expression for a first integral of a certain type. By means of a rational change of variable, we obtain the homogenous second-order linear differential equation: $A_2(x)w''(x) + A_1(x)w'(x) + A_0(x)w(x) = 0$, whose coefficients are polynomials, corresponding to a planar polynomial differential system. We prove that this system has an invariant curve for each arbitrary non-null solution $w(x)$ of the second-order ordinary differential equation, which, in the case $w(x)$ is a polynomial, gives rise to an invariant algebraic curve. In addition, we give an explicit expression of a first integral for the system constructed from two independent solutions of the second order ordinary differential equation. This first integral is not, in general, a Liouvillian function. The inverse integrating factor of the system (1) which is associated to the aforementioned second-order linear differential equation, takes the form $V(x, y) = q(x)(w'(x) - g(x, y)w(x))^2$ where $q(x) = A_2(x)\exp\left\{\int A_1(x)\big/A_2(x)\,\mathrm{d}x\right\}$, $g(x, y)$ is a fixed rational function and $w(x)$ is a non-null solution of the second-order linear differential equation.

Moreover, in the work [8] we also consider first-order linear differential equations: $A_1(x)w'(x)+A_0(x)w(x) = 0$ with polynomial coefficients and analogous results are obtained. The inverse integrating factor of system (1) which is associated to this first-order linear differential equation, takes the form $V(x, y) = A_1(x)g(x, y)(w(x) - a(x, y))$ where $a(x, y)$ is a function defined in terms of $A_1(x)$ and $A_0(x)$, $g(x, y)$ is fixed rational functions and $w(x)$ is a non-null solution of the first-order linear differential equation.

Hence, in the work [8], we give families of systems which, by construction, linearize, because their corresponding inverse integrating factors are obtained in terms of the solutions of a linear differential equation. Moreover, the given families are very general since they come from any rational change of variables. The present work comes as a reciprocal of the work [8], since we look for systems which can be linearized, in the sense of definition 1.

The goal of this work is to demonstrate an algorithm to detect when a system is integrable (either inside the Liouvillian class or not) by means of a linearization process. That is, we target to find systems whose integrability passes through the solutions of a linear differential equation. We proceed by giving and explaining several examples which illustrate this process.

The examples that we study suggest that the integrability by linearization of a polynomial system (1) reduces to solve linear differential equations of order 2 or it falls into the Darboux theory of integrability.

The studied examples also give rise to the following questions: when a system is linearizable (in the sense of definition 1) with a linear differential equation of order 1, does a rational change of variables always exist which transforms the system to an orbital equation which is linear? In the same way, we can also ask whether when a system is linearizable (in the sense of definition 1) with a linear differential equation of order 2, a rational change of variables always exists which transforms the system to an orbital equation of Riccati type?

For the families of systems studied in [8], the answer to the previous two questions is affirmative.

The question of linearizability has attracted many authors since the transformation of an ordinary differential equation or a partial differential equation of any order by means of several differential–algebraic manipulations to a linear differential equation, gives in general the solution of the first, nonlinear problem. Moreover, ordinary differential equations which linearize come naturally with some physical applications, see [7] and the references therein. In the work [7], the question of which ordinary differential equations (of any order) linearize upon differentiation is addressed and some sufficient conditions on the form of the equation are given. However, these sufficient conditions are very restrictive over the equations and only very special particular equations can satisfy them. We only consider ordinary differential equations of first order, that is systems of the form (1) and we study several differential–algebraic manipulations so as to get a linear equation which characterizes its integrability.

We use two different methods to exhibit that a system is linearizable: *equivalence* and *compatibility*. Both methods start in the same way. We consider system (1) and think of it as polynomial in one variable, for instance y. Then, we take a polynomial in the variable y of a certain fixed degree and with arbitrary coefficients, which are functions of the variable x, $\sum_{i=0}^{M} h_i(x) y^i$, and we impose it to be an inverse integrating factor of the corresponding system (1). This condition gives rise to a system of linear differential equations on the coefficients $h_i(x)$. In general, this system of linear differential equations is overly determined. Several conditions on system (1) can make this system compatible and the way to choose these conditions is what distinguishes between the both methods.

The *compatibility method* is the grossed one between the two: we consider the system of linear differential equations with variables $h_i(x)$ and we uncouple the variables by means of differentiation and resultants. We end up with an algebraic–differential condition on system (1). Although this method gives all the possible choices for system (1) to have an inverse integrating factor of the prescribed form, it is usually too overwhelming to be carried out.

On the other hand, we can consider the *equivalence method*. This method is wiser and consists of uncoupling the system of linear differential equations with variables $h_i(x)$, avoiding differentiation and resultants, and only uses substitution at each step. We end up with a number of linear differential equations of certain order and of only one variable, say $h_0(x)$, and we make these equations equivalent, that is, we impose them to be the same equation and/or to be identically null for some of them. This method again gives certain particular conditions on system (1) which are, usually, easy to satisfy.

We note that the conditions given by the equivalence method are also contained in the conditions given by the compatibility method but their determination is much easier when the equivalence is involved.

The method of equivalence gives rise to a linear ordinary differential equation of a certain order ℓ for one variable, which can be any $h_i(x)$ in the expression $\sum_{i=0}^{M} h_i(x) y^i$. Each solution of this ordinary differential equation gives, by substitution(now in the reverse manner), an inverse integrating factor for system (1). We would like to know the possible values of ℓ, that is, we ask whether we can linearize systems (1) by means of a linear ordinary differential equation of any order ℓ, with $\ell \geqslant 0$. We aim to know the values of the order ℓ corresponding to a linear differential equation such that each of its solutions cannot be expressed as a polynomial on the solutions of an equation of order lower than ℓ. The following result shows that the order of such a linear differential equation is at most 2.

Theorem 2. *We assume that system (1) has an inverse integrating factor of the form:*

$$V(x, y) = \sum_{i=0}^{M} h_i(x) y^i, \tag{2}$$

where M is a nonnegative integer number and $h_i(x)$ are analytic functions in $x, i = 0, 1, \ldots, M$. We assume that the functions $h_i(x)$, for $i = 1, 2, \ldots, M$, are polynomials in $h_0(x)$ and its derivatives, and that $h_0(x)$ satisfies a linear differential equation of order ℓ, with ℓ a nonnegative integer, whose solutions cannot be algebraically expressed in terms of the solutions of an equation of order lower than ℓ. Then, $\ell \leqslant 2$.

Proof. We know that a linear ordinary differential equation of order ℓ has a fundamental set of solutions with cardinal ℓ. That is, there are ℓ linearly independent solutions to the equation. Assume that $\ell \geqslant 3$ and let $V_1(x, y)$, $V_2(x, y)$ and $V_3(x, y)$ be three inverse integrating factors each one constructed by using one of these linearly independent solutions through the expression (2). The quotients of two of them then give first integrals of system (1): $H_1(x, y) = V_1(x, y)/V_3(x, y)$ and $H_2(x, y) = V_2(x, y)/V_3(x, y)$. These two first integrals need to be functionally dependent since any first integral of a planar differential system like (1) is a function of another one. We now show that, in fact, H_1 and H_2 are algebraically dependent, that is, there exists a polynomial with real coefficients $P(z_1, z_2)$ such that $P(H_1, H_2) \equiv 0$. We consider the level curves of each H_i: $V_i(x, y) - c_i V_3(x, y) = 0$, with $i = 1, 2$, which are two polynomials in y because each $V_i(x, y)$, $i = 1, 2, 3$, is a polynomial in y. Let us take the resultant of the polynomials $V_1(x, y) - c_1 V_3(x, y)$ and $V_2(x, y) - c_2 V_3(x, y)$ with respect to y and we denote it by $R(c_1, c_2, x)$. We remark that this resultant is a polynomial in c_1 and c_2 and it factorizes as $R(c_1, c_2, x) = P(c_1, c_2) R_0(c_1, c_2, x)$ (as we shall see) where $P(c_1, c_2)$ and $R_0(c_1, c_2, x)$ are polynomials in c_1 and c_2. This factorization of $R(c_1, c_2, x)$ is deduced by the fact that each y-root of $V_1(x, y) - c_1 V_3(x, y) = 0$, for a fixed c_1, needs to correspond to a value of c_2 such that the whole y-root is contained in $V_2(x, y) - c_2 V_3(x, y) = 0$. That is, for a fixed c_1 and y-root of $V_1(x, y) - c_1 V_3(x, y) = 0$, there exists a value of c_2 such that this y-root is completely contained in $V_2(x, y) - c_2 V_3(x, y) = 0$. Therefore, we have encountered a polynomial $P(c_1, c_2)$ which relates the two first integrals in the desired way.

Let us call $S_i(x)$ the solution of the linear ordinary differential equation which gives the inverse integrating factor $V_i(x, y)$, $i = 1, 2, 3$. Since $P(H_1, H_2) \equiv 0$, we deduce that there exists a homogeneous polynomial with real coefficients such that $p(S_1, S_2, S_3) \equiv 0$. To deal with this polynomial p, we put $y = 0$ in the expression of $P(H_1, H_2)$ and take a common denominator. The existence of this polynomial p implies that S_3 can be expressed algebraically in terms of S_1 and S_2. We remark that any two given functions S_1 and S_2 satisfy the linear homogeneous ordinary differential equation of second order:

$$\det \begin{vmatrix} w''(x) & w'(x) & w(x) \\ S_1''(x) & S_1'(x) & S_1(x) \\ S_2''(x) & S_2'(x) & S_2(x) \end{vmatrix} = 0.$$

Thus, the function S_3 is algebraically expressed in terms of the solutions of an equation of second order.

We conclude that any inverse integrating factor which shows the linearizability of system (1) through a linear differential equation of order ℓ with $\ell \geqslant 3$, can be expressed in such a way that the linearizability of the system is given through a linear differential equation of, at most, second order. □

In order to make more precise the given notion of linearization of a system (1), we include several examples of this phenomenon and we use the described methods of equivalence or compatibility.

2. Examples

2.1. Automatically linearizable systems

In this section we describe examples of systems for which we find an inverse integrating factor constructed by means of the solutions of a linear differential equation, that is, we describe examples of linearizability. We do not need to impose any condition on the system to ensure its linearizability, that is, the system of linear differential equations on the functions $h_i(x)$ for which $V(x, y) = \sum_{i=0}^{M} h_i(x) y^i$ is an inverse integrating factor is not overly determined.

Example 1. Let us consider systems of the form (1) with:

$$P(x, y) = -y, \qquad Q(x, y) = \sum_{i=0}^{m} g_{2i}(x) y^{2i}, \tag{3}$$

where $g_{2i}(x)$ are analytic functions and m is an integer number with $m \geqslant 0$. In this section we consider only the case in which $m = 2$.

We look for an inverse integrating factor $V(x, y)$ which is a polynomial in y of the same degree as (3) and of the form:

$$V(x, y) = \sum_{i=0}^{m} h_{2i}(x) y^{2i},$$

where $h_{2i}(x)$ are suitable functions which will satisfy linear differential equations. Our goal is to impose such a function $V(x, y)$ as an inverse integrating factor for system (3) and deduce the relations on the functions $g_{2i}(x)$ to accomplish it.

The case $m = 0$ is easily integrable since the corresponding orbital equation has separate variables. Let us explicit the computations made when $m = 2$. We have the system:

$$\dot{x} = -y, \qquad \dot{y} = g_0(x) + g_2(x) y^2 + g_4(x) y^4, \tag{4}$$

and we look for an inverse integrating factor of the form $V(x, y) = h_0(x) + h_2(x) y^2 + h_4(x) y^4$. We impose the following relation to be satisfied:

$$P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \text{div} \cdot V,$$

where $\text{div} = 2g_2(x) y + 4g_4(x) y^3$. The previous partial differential equation can be arranged in powers of y and we get that three relations among the functions $h_{2i}(x)$ need to be satisfied, corresponding to the powers y^5, y^3 and y. We remark that we have three relations and three functions to satisfy them. This is an exceptional case since the number of equations and the number of variables coincide. These relations read for:

$$\begin{aligned} &h_4'(x) - 2g_2(x) h_4(x) + 2g_4(x) h_2(x) = 0, \\ &h_2'(x) - 4g_0(x) h_4(x) + 4g_4(x) h_0(x) = 0, \\ &h_0'(x) - 2g_0(x) h_2(x) + 2g_2(x) h_0(x) = 0. \end{aligned} \tag{5}$$

We can deduce the values of the functions $h_2(x)$ and $h_0(x)$, for instance, from the first two equations (5). The third equation in (5) gives a third-order linear differential equation for $h_4(x)$ which is:

$$g_4^2 h_4''' - 3g_4 g_4' h_4'' + \left(-4g_2^2 g_4^2 + 16 g_0 g_4^3 - 4g_4^2 g_2' + 4g_2 g_4 g_4' + 3g_4'^2 - g_4 g_4''\right) h_4' \\ + 2((4(g_4 g_0' - g_0 g_4') - 2g_2 g_2' - g_2'')g_4^2 \\ + (2g_2^2 + 3g_2')g_4 g_4' - 3g_2 g_4'^2 + g_2 g_4 g_4'')h_4 = 0, \tag{6}$$

where we avoid writing each as a function of x (and hereafter also when need arises) and thus simplifying the notation.

We have obtained a third-order linear differential equation whose solutions give rise to an inverse integrating factor for system (3) with $m = 2$. We note that we did not need to impose any restriction on the functions $g_{2i}(x)$, $i = 0, 1, 2$, so as to linearize the system. System (3) with $m = 2$, that is system (4), is always linearizable and we have been able to deduce this fact using our method.

Although we have obtained a third-order linear differential equation for $h_4(x)$, we will see that this function is related to a second-order linear ordinary differential equation, as the statement of theorem 2 establishes. Let us impose an inverse integrating factor for system (4) of the form $V(x, y) = (\tilde{h}_0(x) + \tilde{h}_2(x)y^2)^2$, where $\tilde{h}_0(x)$ and $\tilde{h}_2(x)$ are suitable functions. Repeating the same computations as before, we get that the relation corresponding to the power y^5 in the equation of inverse integrating factor gives that $\tilde{h}_0(x) = (g_2(x)\tilde{h}_2(x) - \tilde{h}_2'(x))/(2g_4(x))$ and the other two relations (corresponding to y^3 and y^1) are equal and give the following second order ordinary differential equation for $\tilde{h}_2(x)$:

$$g_4 \tilde{h}_2'' - g_4' \tilde{h}_2' + \left(g_2 g_4' - g_2' g_4 + 4 g_0 g_4^2 - g_2^2 g_4\right)\tilde{h}_2 = 0. \tag{7}$$

The fact that the two relations, corresponding to y^3 and y^1, are equal is not expected and this equality confirms that system (4) is linearizable.

Actually, straightforward computations show that if we denote by $A(x)$ and $B(x)$ two independent solutions of the second order equation (7), then a fundamental set of solutions of (6) is: $A^2(x)$, $A(x)B(x)$ and $B^2(x)$. Hence, the third-order linear ordinary differential equation (6) is, in fact, reducible to a second order ordinary differential equation. This is an example of the result stated in Theorem 2. We note that, in any case, we have linearized system (4). The reduction of order passes, in this case, through the change $h_4(x) = (\tilde{h}_2(x))^2$ which transforms equation (6) in a nonlinear differential equation which is compatible with (7).

We make it precise that when we have linearized system (3) with $m = 2$, that is system (4), we have obtained a third-order linear differential equation (6), but this equation was not necessary since an inverse integrating factor can be obtained through a second-order linear ordinary differential equation, as theorem 2 states.

We consider the algebraic change of variables $y \mapsto z$ with $z = \tilde{h}_0(x) + \tilde{h}_2(x)y^2$, where $\tilde{h}_i$, $i = 0, 2$ are the functions which define the inverse integrating factor $V(x, y) = (\tilde{h}_0(x) + \tilde{h}_2(x)y^2)^2$ which we have already dealt with. This change of variables applied to system (4) gives the following orbital equation:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{z(2zg_4(x) + \tilde{h}_2'(x))}{\tilde{h}_2(x)},$$

which is of Riccati type.

An algebraic change applied to system (4) which gives an orbital equation of Riccati type is not unique. We note that the algebraic change of variables $y \mapsto u$ with $u = y^2$ applied to system (4) gives the following orbital equation:

$$\frac{du}{dx} = -2(g_0(x) + g_2(x)u + g_4(x)u^2), \tag{8}$$

which is also of Riccati type. The linearizable systems studied in the examples show the existence of a rational change of variables which transforms the system to an orbital equation of Riccati type, as a counterpart. We do not look for changes of variables but for linearizability.

It is well-known the equivalence between ordinary differential equations of Riccati type and second-order linear differential equations. For instance, by the change $u \mapsto w$ with $dw/dx = 2g_4(x)u(x)w$ we have that the Riccati equation (8) is equivalent to the following second-order linear differential equation for $w(x)$:

$$g_4(x)w''(x) + (2g_2(x)g_4(x) - g_4'(x))w'(x) + 4g_0(x)g_4^2(x)w(x) = 0. \tag{9}$$

This family of systems also appears as a particular case of the systems described in [8] by means of the algebraic change of variables $y \mapsto u$ with $u = y^2$, which leads it to the orbital equation of Riccati type (8). Following the ideas described in [8], if we consider equation (7) and we perform the change $\tilde{h}_2(x) = \exp\left\{\int_0^x g_2(s)\, ds\right\} w(x)$, we obtain the linear differential equation (9). We would like to remark that the linearizability process does not pass through the second-order linear differential equation (9), which is equivalent to the orbital equation (8). The linearizability of system (4) is concerned with the second-order linear differential equation (7) whose solutions define an inverse integrating factor for the system. However, to impose that system (4) has an inverse integrating factor constructed with the solutions of a linear differential equation seems to imply that the orbital equation associated to the system is equivalent to a linear differential equation by means of an algebraic change of variables. All the examples that we present in this work confirm this implication although the involved linear differential equations are different and come from different sources.

2.2. Linearizability by equivalence

In this section we describe several examples of systems of the form (1) which, under certain restrictions, are linearizable. We determine these restrictions by using the equivalence method.

Example 2. We consider system (3) with $m \geqslant 3$ and explain a process which encounters a linearizable subfamily. Let us describe the computations for system (3) with $m = 3$. In fact, for $m = 1$ and for any $m \geqslant 3$ the discussion is analogous. The only cases which are different are $m = 0$ (separated variables) and $m = 2$ (second-order linear differential equation) which have already been treated in example 1. We consider the system:

$$\dot{x} = -y, \qquad \dot{y} = g_0(x) + g_2(x)y^2 + g_4(x)y^4 + g_6(x)y^6,$$

and we take a function of the form $V(x, y) = h_0(x) + h_2(x)y^2 + h_4(x)y^4 + h_6(x)y^6$. By imposing it to be an inverse integrating factor and equating with the same powers of y, we get five relations which need to be satisfied and which correspond to the coefficients of y, y^3, y^5, y^7 and y^9. From the coefficient of y^9 we compute $h_4(x)$ in terms of $g_{2i}(x)$ and $h_6(x)$. In the same way, from the coefficient of y^7 we compute $h_2(x)$ and from the coefficient of y^5, we compute $h_0(x)$ in terms of $g_{2i}(x)$ and $h_6(x)$. We are left with two linear differential equations of second order for $h_6(x)$ which read for:

$$\frac{h_6''}{4g_6}+\left(\frac{g_4^2}{3g_6^2}-\frac{g_2}{g_6}-\frac{g_6'}{4g_6^2}\right)h_6'+\left[\frac{1}{3g_6}\left(\frac{g_4^2}{g_6}\right)'-\frac{g_4^2g_6'}{3g_6^3}-\left(\frac{g_2}{g_6}\right)'\right]h_6=0,$$

$$\frac{g_4h_6''}{12g_6}+\left(\frac{1}{3}\left(\frac{g_4}{g_6}\right)'-\frac{g_4'}{12g_6}+\frac{g_2g_4}{6g_6}-\frac{3g_0}{2}\right)h_6'$$
$$+\left(\frac{g_4g_6'^2}{2g_6^3}-\frac{g_4'g_6'}{2g_6^2}+\frac{g_2}{3}\left(\frac{g_4}{g_6}\right)'+\frac{g_0g_6'}{g_6}-g_0'\right)h_6=0.$$

We apply the equivalence method to these two linear differential equations, that is, we impose the values of $g_{2i}(x)$ to make them the same equation. Astonishingly, we only need to impose the condition:

$$g_0(x)=\frac{g_2(x)g_4(x)}{3g_6(x)}-\frac{2}{27}\frac{g_4(x)^3}{g_6(x)^2}+\frac{1}{6}\left(\frac{g_4(x)}{g_6(x)}\right)',$$

so as to get only one second-order linear differential equation for $h_6(x)$:

$$3g_6h_6''+\left(4g_4^2-12g_2g_6-3g_6'\right)h_6'+\left(8g_4g_4'-\frac{8g_4^2g_6'}{g_6}+12g_2g_6'-12g_6g_2'\right)h_6=0. \qquad (10)$$

Hence, we have a family of systems of the form (3) which linearize. At this point we have met our target since we have encountered a family of systems whose integrability passes through the solution of a second-order linear differential equation.

We now describe another unexpected phenomenon which occurs in this family. The second-order linear equation for h_6 can be reduced to a linear equation of order 1. We remark that Theorem 2 does not apply because equation (10) is of order 2. We consider system (3) with $m = 3$ and the described value of the function $g_0(x)$. We impose a function of the form $V(x, y) = (\tilde{h}_0(x)+\tilde{h}_2(x)y^2)^3$ to be an inverse integrating factor. We get that $\tilde{h}_0 = g_4\tilde{h}_2/(3g_6)$ and obtain only one linear homogeneous differential equation of order 1 for the function $\tilde{h}_2(x)$ which is $9g_6\tilde{h}_2'+4\left(g_4^2-3g_2g_6\right)\tilde{h}_2=0$. Therefore, we have that system (3) with $m = 3$ and the described value of $g_0(x)$ linearizes. We remark that the final equation for $\tilde{h}_2(x)$ in the case $m = 3$ is of order 1 whereas the final equation for $\tilde{h}_2(x)$ in the case $m = 2$ is of order 2.

Moreover, this family of systems also appears as a particular case of the work [8]. We consider the algebraic change of variables $y \mapsto z$ with $z = \tilde{h}_0(x) + \tilde{h}_2(x)y^2$, where $\tilde{h}_i, i = 0, 2$ are the functions which define the encountered inverse integrating factor $V(x, y) = (\tilde{h}_0(x) + \tilde{h}_2(x)y^2)^3$. This change of variables applied to the considered family of systems gives the following orbital equation:

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{2}{9}z\left(-3g_2(x)+\frac{g_4(x)^2}{g_6(x)}-\frac{9g_6(x)z^2}{\tilde{h}_2(x)^2}\right), \qquad (11)$$

which is a differential equation of Bernoulli type. The change $z \mapsto u$ with $z = 1/\sqrt{u}$ transforms equation (11) to a linear differential equation. We note that by this method we impose a system to be linearized, that is, the other way round of what we obtained in [8], where we started by the linear differential equation and deduced the corresponding system. As far as the examples described in this work indicate, we have observed that when a linearizable system (1) (in the sense of definition 1) admits an inverse integrating factor of the form $V(x, y) = c(x, y)^n$, where n is a positive integer and $c(x, y)$ is a polynomial in y, the algebraic change $y \mapsto z$, where $z = c(x, y)$, applied to the system gives an orbital equation which is of Riccati or Bernoulli type.

The same computations can also be done for the systems (3) with $m = 1$ or $m > 3$, and giving certain values to $m - 2$ of the functions $g_{2i}(x)$, that is we have three arbitrary functions $g_{2i}(x)$ in the expression of $Q(x, y)$, an inverse integrating factor of the form $V(x, y) = (\tilde{h}_0(x) + \tilde{h}_2(x)y^2)^m$ is exhibited, where $\tilde{h}_0(x)$ is expressed in terms of the $g_{2i}(x)$ and $\tilde{h}_2(x)$, and the function $\tilde{h}_2(x)$ is the solution of a linear homogeneous ordinary differential equation of order 1. In the particular case where the linearizable system (3) with $m \geqslant 3$ is a polynomial differential system, we have that the encountered inverse integrating factor is of Darboux type. We remark that in the method of linearization we are not seeking for invariant algebraic curves but we have obtained them by construction.

When we have studied systems of the form (3) with $m = 2$ we have exhibited a system in which the linearization is automatically met. When we have described the systems of the form (3) with $m = 3$, we have given a family of linearizable systems in which the linearization is met by the equivalence method. We also remark that in case system (3) is polynomial, we can encounter non-Liouvillian inverse integrating factors when $m = 2$, that is, when the system is of degree 4 in variable y. When $m \geqslant 3$ in a polynomial system (3), the linearization process gives a Darboux inverse integrating factor and, thus, a Liouvillian first integral. This fact implies that the encountered second-order linear differential equation (10) only has elementary solutions.

Example 3. The following family of systems is a more general case and we use equivalence method so as to deal with this subfamily of linearizable systems. Let us consider the system:

$$\begin{aligned} \dot{x} &= c_0(x) + c_2(x)y^2, \\ \dot{y} &= d_0(x) + d_1(x)y + d_2(x)y^2 + d_3(x)y^3 + d_4(x)y^4 + d_5(x)y^5 + d_6(x)y^6, \end{aligned} \tag{12}$$

where $c_i(x)$ and $d_i(x)$ are analytic functions with $c_2(x) \cdot d_6(x) \not\equiv 0$. We start by imposing an inverse integrating factor of the form

$$V(x, y) = \sum_{i=0}^{6} h_i(x)y^i,$$

which is a polynomial in y of the same degree as system (12). We substitute this expression of V in the partial differential equation that must be satisfied to be an inverse integrating factor and we obtain that a certain polynomial in y of degree 10 must be identically zero. From the coefficients in y of this equation of degrees from 10 to 5 we deduce the values of $h_i(x)$ with $i = 0, 1, \ldots, 5$. The rest of the coefficients give five linear differential equations for $h_6(x)$. Two of them are of order 3 and the rest of the order 2. We impose the three equations of order 2 to be identically null, which give the following conditions on system (12):

$$\begin{aligned} &d_5(x) = 0, \qquad d_4(x) = \frac{6c_0(x)d_6(x)}{c_2(x)}, \qquad d_2(x) = \frac{9c_0(x)^2 d_6(x)}{c_2(x)^2}, \\ &d_1(x) = \frac{3c_0(x)d_3(x)}{c_2(x)} - c_2(x)\left(\frac{c_0(x)}{c_2(x)}\right)'. \end{aligned} \tag{13}$$

Under these conditions we get that the three equations of order 2 are identically null and, surprisingly, the two equations of third order are equal. The fact that under this small number of restrictions, the involved linear differential equations become equivalent is unexpected and confirms the hidden structure of the linearizability process. This subfamily of systems (12) is linearizable.

We remark that this is not the only way to proceed so as to get linearizability. We could also have imposed the two equations of third order to be identically null and then use equivalence with the second order equations. We have only presented one of the possible cases that we dealt with.

We know that the considered third-order linear differential equation can be reduced because its solutions cannot be functionally independent. Let us consider an inverse integrating factor for system (12) of the form $V(x, y) = (\tilde{h}_3(x)y^3+\tilde{h}_2(x)y^2+\tilde{h}_1(x)y+\tilde{h}_0(x))^2$. Analogous computations give that, under the same conditions, the determination of $V(x, y)$ comes from the solution of the following second-order linear differential equation in $\tilde{h}_3(x)$:

$$\frac{2c_2^2}{3d_6}\tilde{h}_3'' - \frac{2c_2^2d_6'}{3d_6^2}\tilde{h}_3' + \left(\frac{c_2'^2}{6d_6} - \frac{c_2c_2''}{3d_6} + \frac{c_2c_2'd_6'}{3d_6^2} + c_2\left(\frac{d_3}{d_6}\right)' - \frac{3d_3^2}{2d_6} + 6d_0\right)\tilde{h}_3 = 0.$$

We note that the change of variables $y \mapsto u$ with

$$c_2(x)^2u + 3d_6(x)y(3c_0(x) + c_2(x)y^2) = 0$$

transforms the orbital equation associated to system (12), with the values imposed in (13), to a Riccati equation. We have that system (12), with the values described in (13), is a particular case of the results given in [8]. As before, when we apply the linearization process we do not look for systems which come from Riccati equations via a change of variables, but we obtain such systems as a counterpart.

2.3. *Linearizability by compatibility*

In this section we describe several examples of systems of the form (1) which, under certain restrictions, are linearizable, in the sense of definition 1. We deal with these restrictions by applying the compatibility method.

Example 4. Let us consider the following system which appears in the work [4]. The system

$$\dot{x} = y(-1 + 2\rho^2(x^2 - y^2)), \qquad \dot{y} = x + \rho x^2 + \rho y^2 + 4\rho^2xy^2, \tag{14}$$

where $\rho \in \mathbb{R}$, has the inverse integrating factor $V(x, y) = (x^2 + y^2)^2(1 + 2\rho x + \rho^2(x^2 + y^2))$. We next deal with this inverse integrating factor using the compatibility method. This is the first example in which the compatibility method appears.

We consider system (14) and we impose an inverse integrating factor of the form $V(x, y) = h_0(x) + h_2(x)y^2 + h_4(x)y^4 + h_6(x)y^6$. We equate to zero the coefficients of the powers of y in the relation that makes $V(x, y)$ an inverse integrating factor. We have five equations corresponding to the coefficients of y^j for $j = 1, 3, 5, 7, 9$ which read for:

$$\begin{aligned}
&2x(1 + \rho x)h_2(x) = (1 - 2\rho^2x^2)h_0'(x) + 2\rho(1 + 6\rho x)h_0(x),\\
&4x(1 + \rho x)h_4(x) - 2\rho^2h_0'(x) = (1 - 2\rho^2x^2)h_2'(x) + 4\rho^2xh_2(x),\\
&6x(1 + \rho x)h_6(x) - 2\rho^2h_2'(x) = (1 - 2\rho^2x^2)h_4'(x) - 2\rho(1 + 2\rho x)h_4(x),\\
&2\rho^2h_4'(x) = -(1 - 2\rho^2x^2)h_6'(x) + 4\rho(1 + 3\rho x)h_6(x),\\
&2\rho^2h_6'(x) = 0.
\end{aligned}$$

We remark that we have five linear differential equations for the four functions $h_0(x), h_2(x), h_4(x)$ and $h_6(x)$. This system of linear differential equations is shown to be overly determined. In the following We proceed in several ways but with no loss of generality in any of them. We can, for instance, take the equations from the last one to the first one in the order they have been written. We solve them, leaving an arbitrary constant at each step. We end up with several algebraic relations for these constants which mark their value and from which we get the inverse integrating factor for the previously described polynomial. The compatibility of these relations is obtained by an adequate choice of the constants of integration. Another way to study this system of linear differential equations is to take them in the order they have been written. From the first one we equate $h_2(x)$ and we substitute it in the rest of equations. From the second equation, we equate $h_4(x)$ and we again substitute it in the rest of equations and from the third equation we get $h_6(x)$. We end up with two linear differential equations of fourth order for $h_0(x)$ and we make them compatible. The compatibility process goes as follows: we consider the two linear differential equations of fourth order for $h_0(x)$ and we make a linear combination of them so as to get a linear differential equation of third order. We derive this third-order linear differential equation and we combine it with one of the previously considered linear differential equations of fourth order for $h_0(x)$. We have two third-order linear differential equations for $h_0(x)$ and we combine them so as to get a linear differential equation of second order:

$$x(1+\rho x)(1-7\rho^2x^2-6\rho^3x^3+13\rho^4x^4+24\rho^5x^5+12\rho^6x^6)h_0''(x)-(3+4\rho x-27\rho^2x^2 \\ -62\rho^3x^3+33\rho^4x^4+224\rho^5x^5+252\rho^6x^6+96\rho^7x^7)h_0'(x)+2\rho(-3-12\rho x \\ -17\rho^2x^2+44\rho^3x^3+189\rho^4x^4+240\rho^5x^5+108\rho^6x^6)h_0(x)=0.$$

We derive it and combine with one of the previously considered equations of third order, so as to get another linear differential equation of second order:

$$x(1+\rho x)(-1-4\rho x+30\rho^2x^2+116\rho^3x^3-176\rho^4x^4-1096\rho^5x^5-379\rho^6x^6+3914\rho^7x^7 \\ +5454\rho^8x^8-2908\rho^9x^9-11756\rho^{10}x^{10}-7480\rho^{11}x^{11}+2832\rho^{12}x^{12} \\ +5184\rho^{13}x^{13}+1728\rho^{14}x^{14})h_0''(x)-(-3-16\rho x+80\rho^2x^2+524\rho^3x^3 \\ -108\rho^4x^4-5104\rho^5x^5-7279\rho^6x^6+15162\rho^7x^7+49126\rho^8x^8+21124\rho^9x^9 \\ -79964\rho^{10}x^{10}-125496\rho^{11}x^{11}-43376\rho^{12}x^{12}+48576\rho^{13}x^{13} \\ +50112\rho^{14}x^{14}+13824\rho^{15}x^{15})h_0'(x)+2\rho(3+24\rho x+28\rho^2x^2-386\rho^3x^3 \\ -1790\rho^4x^4-1246\rho^5x^5+10269\rho^6x^6+28150\rho^7x^7+11134\rho^8x^8 \\ -60804\rho^9x^9-106460\rho^{10}x^{10}-47832\rho^{11}x^{11}+39312\rho^{12}x^{12}+50112\rho^{13}x^{13} \\ +15552\rho^{14}x^{14})h_0(x)=0.$$

We have at this step two linear differential equations of second order for $h_0(x)$, which we combine so as to get a first-order linear differential equation for $h_0(x)$:

$$x(1+\rho x)h_0'(x)=2(2+3\rho x)h_0(x).$$

We derive it and we obtain a second-order linear differential equation for $h_0(x)$ which, combined with one of the previous gives rise to a first-order linear differential equation. The two first-order linear differential equations for $h_0(x)$ turn out to be the same. If this was not the case, we would combine them and we would obtain a compatibility condition on the coefficients of the system. In our case, we solve this first-order linear differential equation for $h_0(x)$. This value of $h_0(x)$ is the value which makes compatible the two fourth-order linear differential equations from which we started the process. The only possible common solution of these two fourth-order linear differential equations is $h_0(x)=x^4(1+\rho x)^2$ (modulus a multiplicative constant). We observe that this $h_0(x)$ univocally determines the previously described inverse integrating factor.

This example suggests that the integrability by linearization of a polynomial system (1) reduces to solve linear differential equations of order 2 or it falls into the Darboux theory of integrability. We remark that any Darboux inverse integrating factor which is a polynomial in the variable y is encountered by our linearization process: either by the equivalence method or by the compatibility method.

We note that when applying the linearization process to system (14) we obtain a Darboux inverse integrating factor and, thus, invariant algebraic curves of the systems as a counterpart.

Example 5. In this example we address the question whether the family (14) can be embedded in a linearizable family and it can be seen as a particular case of linearization with equivalence. We can think of system (14) as a particular case of the following family of systems:

$$\dot{x} = y(g_1(x) + g_2(x)y^2), \qquad \dot{y} = g_3(x) + g_4(x)y^2, \tag{15}$$

where $g_i(x)$ are arbitrary functions. In the case $g_2(x) \equiv 0$, we have that system (15) coincides with system (3) with $m = 2$ after a time-rescaling, which has already been studied in the first example. Therefore, we can assume, without loss of generality, that $g_2(x) \equiv 1$. We note that the family (15) is reversible by the change $(x, y, t) \mapsto (x, -y, -t)$ and, thus, any ansatz of an inverse integrating factor must be even in y. Let us consider an inverse integrating factor of the form:

$$V(x, y) = h_0(x) + h_2(x)y^2 + h_4(x)y^4 + h_6(x)y^6.$$

Assuming this to be an inverse integrating factor gives a polynomial in y. We observe that this relation is an odd polynomial in y of degree 9 which is odd. The functions $h_i(x)$ must vanish each one of the coefficients of this polynomial in y. From the coefficient of y^9 of this polynomial we deduce that $h_6(x) = k_6$, with k_6 a constant value which we assume to be nonzero. From the coefficient of y, we deduce the value of $h_2(x)$, and from the coefficient of y^3, the value of $h_4(x)$. The coefficients of y^5 and y^7 give rise to two linear differential equations of order 3 for $h_0(x)$.

When applying equivalence to these two equations, that is, imposing them to be the same equation, we deduce the following conditions:

$$g_3(x) = \frac{1}{4}\left(2g_1(x)g_4(x) - g_1(x)g_1'(x)\right), \qquad g_4(x) = -\frac{g_1'(x)}{2}.$$

In this case, the orbital equation associated to system (15) is of separated variables. We observe that the linearization process for this example leads to an ordinary differential equation with separated variables.

Another way to study the possible linearizability of system (15) is to impose that the two linear equations of third order have a nonzero common solution, that is, to impose compatibility. The computations for the compatibility method for these two equations of third order carries long calculations and many complex cases. If we apply compatibility to the two linear differential equations of order 3 for $h_0(x)$, we obtain several conditions. The conditions obtained in the equivalence case are reencountered now. Moreover, we obtain two additional, and very complicated, conditions on the functions $g_i(x)$ to have compatibility. One of these two conditions is the one satisfied by system (14) as a particular case. We have seen that system (14) cannot be seen as a particular case of a linearizable family of systems (15) by the equivalence method.

Example 6. We consider a planar differential system of the form:

$$\dot{x} = g_0(x) + g_1(x)y + g_2(x)y^2, \qquad \dot{y} = g_3(x)y + g_4(x)y^2, \tag{16}$$

where $g_i(x), i = 0, 1, 2, 3, 4$, are arbitrary functions. We remark that this system has $y = 0$ as an invariant algebraic curve and we propose an inverse integrating factor which contains this information. We now give conditions on the functions $g_i(x), i = 0, 1, 2, 3, 4$, such that the system has an inverse integrating factor of the form:

$$V(x, y) = h_1(x)y + h_2(x)y^2 + h_3(x)y^3,$$

where $h_i(x), i = 1, 2, 3$ are suitable functions. The imposition for V to be an inverse integrating factor of system (16) gives rise to a polynomial in y of degree 5. From the coefficients of y^5 and y^1 of this polynomial we deduce that:

$$h_3(x) = k_2 g_2(x), \qquad h_1(x) = k_0 g_0(x),$$

where k_0 and k_2 are arbitrary constants. We end up with only three conditions which involve the functions $g_i(x), i = 0, 1, 2, 3, 4$ and the constants k_0, k_2. Since the function $h_2(x)$ is not concerned, we take $h_2(x) \equiv 0$ for simplicity. The vanishing of these three conditions gives the following planar differential system:

$$\begin{aligned} \dot{x} &= k_2 g_1(x)(g_0(x)^2 + g_0(x)g_1(x)y + g_1(x)^2y^2), \\ \dot{y} &= y(k_0g_0(x) - k_2g_0(x) - k_2yg_1(x))(g_0'(x)g_1(x) - g_0(x)g_1'(x)), \end{aligned} \tag{17}$$

where k_0, k_2 are real constants, and $g_0(x)$ and $g_1(x)$ are analytic functions. This system has the inverse integrating factor $V(x, y) = yg_1(x)(k_0g_0(x)^2 + k_2g_1(x)^2y^2)$. The following rational change of variables $x \mapsto z$ with $z = 1 - k_2 + yg_1(x)/g_0(x)$ transforms the orbital equation associated to the system to the following linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{yz}{(1 - k_2 - z)(k_2 + k_1(1 - k_2 - z)^2)}.$$

2.4. Inverse integrating factors of only one variable

Example 7. In the following example we describe another method of linearization, which consists of imposing conditions to the system so as to obtain an inverse integrating factor depending only on y.

Let us consider the following system:

$$\dot{x} = y + c_1(x) + c_2(x)y^2, \qquad \dot{y} = c_3(x) + c_4(x)y + c_5(x)y^2, \tag{18}$$

where $c_i(x), i = 1, 2, 3, 4, 5$ are arbitrary functions. Let us impose a function $V = V_0(y)$ to be an inverse integrating factor and we look for conditions on $c_i(x), i = 1, 2, 3, 4, 5$, to accomplish this fact. The condition for $V_0(y)$ to be an inverse integrating factor for (18) reads for:

$$(c_3(x) + c_4(x)y + c_5(x)y^2)V_0'(y) = (c_1'(x) + c_4(x) + 2c_5(x)y + c_2'(x)y^2)V_0(y).$$

Imposing that $c_j(x) = k_jc_1'(x)$ for $j = 3, 4, 5$ and $c_2(x) = k_1 + k_2c_1(x)$ with $k_i, i = 1, 2, 3, 4, 5$, arbitrary constants, we get a linear differential equation for $V_0(y)$. We change $c_1(x)$ to $c(x)$ for simplicity of notations and we have that the system:

$$\dot{x} = c(x) + y + (k_1 + k_2c(x))y^2, \qquad \dot{y} = c'(x)(k_3 + k_4y + k_5y^2), \tag{19}$$

linearizes since it has an inverse integrating factor $V = V_0(y)$ which needs to satisfy the linear differential equation: $(k_3 + k_4y + k_5y^2)V_0'(y) = (1 + k_4 + 2k_5y + k_2y^2)V_0(y)$. We note that the function $V_0(y)$ is the exponential of the primitive of a rational function, that is, it is of Darboux type.

We are going to present a change of variables for system (19) which transforms the corresponding orbital equation to a linear differential equation. The following rational change of variables $x \mapsto z$ with $z = c(x) + y + (k_1 + k_2c(x))y^2$ transforms the orbital equation associated to system (19) into:

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{1 + 2k_1y - k_2y^2}{1 + k_2y^2} + \frac{1 + 2k_2k_3y + 2k_2y^2(1 + k_4 + k_5y) + k_2^2y^4}{(1 + k_2y^2)(k_3 + k_4y + k_5y^2)}z,$$

which is linear. Thus, we get that system (19) is a particular case of the families described in [8].

It is evident that linearizable systems of the form (19) of any degree in y can be constructed in an analogous way. The rational change of variables which would transform it to an ordinary differential equation of linear type, and thus relate it to the work [8], is $x \mapsto z$ with $z = P(x, y)$, where $P(x, y)$ is the function defined by $\dot{x} = P(x, y)$.

This example suggests the following question: does an algebraic change of variables always exists that transforms a linearizable system with a Darboux inverse integrating factor of only one variable to a system whose orbital equation is linear?

2.5. Inverse integrating factors of the form $V(x, y) = r(x)h(y)$

Example 8. The following examples of linearizable systems are obtained imposing that the inverse integrating factor is a product of two functions: one in the variable x and the other in the variable y.

Let us consider the system

$$\dot{x} = g_1(x)f_1(y) + g_2(x)f_2(y), \qquad \dot{y} = g_3(x)f_3(y) + g_4(x)f_4(y), \tag{20}$$

where $g_i(x)$ and $f_i(y)$ are arbitrary functions, $i = 1, 2, 3, 4$. We impose this system to have an inverse integrating factor of the form $V(x, y) = r(x)h(y)$. When we substitute this expression in the partial differential equation which defines an inverse integrating factor, we get:

$$\left(f_3(y) + f_4(y)\frac{g_4(x)}{g_3(x)}\right)h'(y) + \left[f_4'(y)\frac{g_4(x)}{g_3(x)} + f_2(y)\left(\frac{g_2(x)r'(x) - g_2'(x)r(x)}{r(x)g_3(x)}\right)\right.$$
$$\left. + f_1(y)\left(\frac{g_1(x)r'(x) - g_1'(x)r(x)}{r(x)g_3(x)}\right)\right]h(y) = 0.$$

We take

$$g_3(x) = -\frac{g_1(x)}{k_0}\left(\frac{g_2(x)}{g_1(x)}\right)', \qquad g_4(x) = -\frac{g_1(x)}{k_0k_1}\left(\frac{g_2(x)}{g_1(x)}\right)', \qquad r(x) = k_2g_1(x),$$

where k_i, $i = 0, 1, 2$ are real constants, and the previous relation reads for:

$$\left(f_3(y) + \frac{f_4(y)}{k_1}\right)h'(y) + \left(k_0f_2(y) - f_3'(y) - \frac{f_4'(y)}{k_1}\right)h(y) = 0,$$

which is a linear ordinary differential equation for $h(y)$. We obtain the following system:

$$\begin{aligned}\dot{x} &= k_0g_1(x)(g_1(x)f_1(y) + g_2(x)f_2(y)),\\ \dot{y} &= (k_1f_3(y) + f_4(y))(g_1'(x)g_2(x) - g_1(x)g_2'(x))/k_1,\end{aligned} \tag{21}$$

where k_0, k_1 are real numbers and $g_1(x)$, $g_2(x)$ and $f_i(y)$, $i = 1, 2, 3, 4$, are arbitrary functions. This system has the inverse integrating factor $V(x, y) = g_1(x)^2h(x)$, where $h(x)$ satisfies the aforementioned linear differential equation of order 1.

The following change of variables $x \mapsto z$ with $z = g_2(x)/g_1(x)$ transforms the orbital equation associated to system (21) into the following linear differential equation:

$$\frac{dz}{dy} = -k_0\frac{f_1(y) + zf_2(y)}{k_1f_3(y) + f_4(y)}.$$

As for the previous example we address the question of the existence of an algebraic change of variables that transforms a linearizable system with a Darboux inverse integrating factor of the form $V(x, y) = r(x)h(y)$ to a system whose orbital equation is linear.

3. More general inverse integrating factors

In the first sections of this work we have provided several examples to exhibit that certain families of systems (1), which are polynomial in the variable y, have an inverse integrating factor which is a polynomial in y. In this section we treat other expressions of an inverse integrating factor which contain the polynomials in y as a subclass. This generalization is done in two steps. We first introduce a real parameter α which allows us to study polynomial inverse integrating factors which are a polynomial in y up to a real power. This real parameter does not involve any change in the linearization process described so forth. The second step is to consider inverse integrating factors which are a power series in y and we end up with a numerable set of linear differential–difference equations.

We remark that the linearization process can also be carried out by imposing a function $V_\alpha(x, y)$ of class $\mathcal{C}^1$ in some open set $\mathcal{U}$ of $\mathbb{R}^2$, non-locally null and which satisfies the following partial differential equation:

$$P\frac{\partial V_\alpha}{\partial x} + Q\frac{\partial V_\alpha}{\partial y} = \alpha\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V_\alpha, \qquad (22)$$

where α is a real number. In the case $\alpha = 1$ we recover the method of linearization described in the previous sections and this real free parameter α gives a generalization for the linearization process which can lead to wider families of linearizable systems. The knowledge of a function $V_\alpha(x, y)$ satisfying this partial differential equations gives that $V = V_\alpha^{1/\alpha}$ is an inverse integrating factor of system (1). In the following, we do not impose a system to have an inverse integrating factor but to have a function $V_\alpha(x, y)$ which satisfies the partial differential equation (22) where α is a real parameter.

In order to illustrate this linearization process we describe an example, where the expression of the function $V_\alpha(x, y)$ is not a polynomial in y but a power series in y, that is,

$$V_\alpha(x, y) = \sum_{n=0}^{\infty} v_n(x)y^n.$$

Let us consider a quadratic polynomial-differential system:

$$\begin{aligned} \dot{x} &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{aligned} \qquad (23)$$

where a_{ij} and b_{ij} are real numbers. We impose that $V_\alpha(x, y) = \sum_{n\geqslant 0} v_n(x)y^n$, satisfies the corresponding partial differential equation (22). We fix a natural number n, and equate the coefficients of y^n in the development of (22) and we get the following linear differential–difference equation for $v_n(x)$. As in the rest of the work, $v_n'(x)$ means the derivative of $v_n(x)$ with respect to x.

$$\begin{aligned} &(a_{00} + a_{10}x + a_{20}x^2)v_n'(x) + (a_{01} + a_{11}x)v_{n-1}'(x) + a_{02}v_{n-2}'(x) \\ &\quad + (n+1)(b_{00} + b_{10}x + b_{20}x^2)v_{n+1}(x) + [(n-\alpha)(b_{01} + b_{11}x) \\ &\quad - \alpha(a_{10} + 2a_{20}x)]v_n(x) + ((n - 2\alpha - 1)b_{02} - \alpha a_{11})v_{n-1}(x) = 0. \end{aligned} \qquad (24)$$

We include several examples of planar systems whose integrability can be determined with a function $V_\alpha(x, y)$ which is a power series in y.

Example 9. We are going to take $v_n(x)$ in recurrence (24), of the form $v_n(x) = q(x)p_n(x)$ where $q(x)$ is a suitable function in x (usually of Darboux type) and $p_n(x)$ is a polynomial in x. In particular, in this example we impose a function of the form $v_n(x) = q(x)\varphi_n H_n(x)$ to be a solution of equation (24), where $q(x)$ is a suitable function, φ_n is a suitable sequence of real numbers and $H_n(x)$ is the Hermite orthogonal polynomial of degree n. For further information about orthogonal polynomials and the identities they satisfy, see for instance [1]. The Hermite orthogonal polynomials satisfy the following two identities:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_n'(x) = 2nH_{n-1}(x). \qquad (25)$$

We substitute the expression $v_n(x) = q(x)\varphi_n H_n(x)$ in (24) and we use the previous identities to simplify it. We have that the following relation must be fulfilled:

$$a_{02}q(x)\varphi_{n-2}H'_{n-2}(x) + A_1(n,x)H_{n-1}(x) + A_2(n,x)H_{n-2}(x) = 0,$$

where the $A_i(n,x)$ are expressions involving the sequence φ_n, the function $q(x)$ and the parameters a_{ij}, b_{ij} and α. Since we have already used all the identities of the Hermite polynomials, we need that the coefficients of $H'_{n-2}(x)$, $H_{n-1}(x)$ and $H_{n-2}(x)$ independently vanish. We impose that $a_{02} = 0$ and the following values make that $A_i(n,x) = 0$ for $i = 1, 2$:

$$\varphi_n = \frac{1}{n!}\left(-\frac{a_{11}}{a_{20}}\right)^{n-1}, \qquad a_{00} = 0, \qquad a_{10} = \frac{a_{01}a_{20}}{a_{11}},$$

$$b_{01} = -\frac{a_{01}a_{20}}{a_{11}}, \qquad b_{11} = -a_{20}, \qquad b_{02} = 0, \qquad \alpha = -1,$$

$$q(x) = \exp\left\{\frac{2(a_{11}b_{10} - a_{01}b_{20})x + a_{11}b_{20}x^2}{a_{20}^2}\right\}(a_{01} + a_{11}x)^{\mu},$$

with $\mu = 2\frac{a_{01}^2 b_{20}}{a_{11}a_{20}^2} + 2\frac{(a_{11}b_{00} - a_{01}b_{10})}{a_{20}^2} - 1$.

We rename the free parameters by $a_{01} = a_0 a_{11}, a_{20} = -a_2 a_{11}, b_{00} = b_0 a_{11}, b_{10} = b_1 a_{11}, b_{20} = b_2 a_{11}$ and a_{11} is taken to be $a_{11} = 1$. We obtain that the quadratic system:

$$\dot{x} = (a_0 + x)(y - a_2 x), \qquad \dot{y} = b_0 + b_1 x + b_2 x^2 + a_2(a_0 + x)y, \tag{26}$$

has the following expression $V_\alpha(x,y)$ which satisfies the partial differential equation (22) with $\alpha = -1$:

$$V_\alpha(x,y) = q(x)\sum_{n\geqslant 0}\varphi_n H_n(x)y^n = q(x)a_2\sum_{n\geqslant 0}\frac{1}{n!}H_n(x)\left(\frac{y}{a_2}\right)^n.$$

The Hermite polynomials have the following generating function:

$$\exp\{2xy - y^2\} = \sum_{n\geqslant 0}\frac{1}{n!}H_n(x)y^n,$$

which also appears in the book [1]. This identity allows us to identify the power series given by $V_\alpha(x,y)$ and we obtain the following inverse integrating factor $V = V_\alpha^{-1}$ for system (26)

$$V_\alpha(x,y) = \exp\left\{\frac{y^2 - 2a_2xy - b_2x^2 + 2(a_0b_2 - b_1)x}{a_2^2}\right\}(a_0 + x)^{\frac{-a_2^2 - 2b_0 + 2a_0b_1 - 2a_0^2b_2}{a_2^2}},$$

which is of Darboux type and not a polynomial in any of the variables x or y.

In this example, we have solved recurrence (24) by imposing it to be compatible with the identities (25). Using the solution of the recurrence, we have encountered the inverse integrating factor of system (26) and we have, therefore, integrated the system.

Example 10. The system $\dot{x} = 1 - x^2$, $\dot{y} = y(x - y)$, has an inverse integrating factor of the form $V = V_\alpha^{1/\alpha}$ with $\alpha = -1/2$ and where:

$$V_\alpha(x,y) = \frac{1}{(1-x^2)^{1/4}}\frac{1 - xy}{1 - 2xy + y^2}.$$

This function $V_\alpha(x,y)$ has been determined using that it satisfies;

$$V_\alpha(x,y) = q(x)\sum_{n=0}^{\infty}T_n(x)y^n,$$

where $q(x) = (1 - x^2)^{-1/4}$ and $T_n(x)$ is the Chebyshev polynomial of first kind and of degree n. This choice makes the recurrence equation (24) to be satisfied in this case. As stated in the book [1], the Chebyshev polynomials of first kind satisfy the following identities:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_n'(x) = 2nT_{n-1}(x) + \frac{n}{n-2}T_{n-2}'(x),$$
$$\frac{1 - xy}{1 - 2xy + y^2} = \sum_{n=0}^{\infty} T_n(x)y^n.$$

Example 11. The system $\dot{x} = 1 - x^2$, $\dot{y} = 1 - xy$, has an inverse integrating factor of the form:

$$V(x, y) = \frac{(1 - x^2)^2}{\sqrt{1 - 2xy + y^2}},$$

which has been determined using that it satisfies;

$$V(x, y) = q(x)\sum_{n=0}^{\infty} P_n(x)y^n,$$

where $q(x) = (1 - x^2)^2$ and $P_n(x)$ is the Legendre polynomial of degree n. As stated in the book [1], the Legendre polynomials satisfy the following identities:

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x),$$
$$P_n'(x) = \frac{n}{x^2 - 1}(xP_n(x) - P_{n-1}(x)),$$
$$\frac{1}{\sqrt{1 - 2xy + y^2}} = \sum_{n=0}^{\infty} P_n(x)y^n.$$

Example 12. The system

$$\begin{aligned} \dot{x} &= 1 + (2a^2 - 7)y^2 + 6y^4 - 2axy(1 + y^2) + 2x^2y^2, \\ \dot{y} &= 2y^2(1 - y^2)(a - xy), \end{aligned} \tag{27}$$

where a is a real parameter, has an inverse integrating factor of the form:

$$V(x, y) = \frac{1}{\sqrt{1 - y^2}} \exp\left\{\frac{2axy - (a^2 + x^2)y^2}{1 - y^2}\right\}.$$

This function $V(x, y)$ has been determined using that it satisfies;

$$V(x, y) = \sum_{n=0}^{\infty} \frac{H_n(a)H_n(x)}{n!2^n}y^n,$$

where $H_n(x)$ is the Hermite polynomial of degree n. The identity

$$\frac{1}{\sqrt{1 - y^2}} \exp\left\{\frac{2axy - (a^2 + x^2)y^2}{1 - y^2}\right\} = \sum_{n=0}^{\infty} \frac{H_n(a)H_n(x)}{n!2^n}y^n,$$

is called *Mehler's—Hermite polynomial formula*, see for instance [2].

These examples suggest that there is a connection between some Darboux inverse integrating factors and orthogonal polynomials.

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